Head and Flow of Ground Water of Variable Density

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Abstract. Fresh-water and environmental-water heads are shown to be useful in studying movement of ground water of variable density, such as in a system of fresh, diffused, and salt water. Fresh-water head at a given point in ground water of variable density is defined as the water level in a well filled with fresh water from that point to a level high enough to balance the existing pressure at the point. Fresh-water heads define hydraulic gradients along a horizontal. An environmental-water head at a given point in ground water of variable density is defined as a fresh-water head reduced by an amount corresponding to the difference of salt mass in fresh water and that in the environmental water between that point and the top of the zone of saturation. Environmental-water heads define hydraulic gradients along a vertical. Vertical and horizontal components of velocity in an anisotropic system with ground water of variable density are computed from hydraulic gradients defined by environmental-water and fresh-water heads, respectively, and from appropriate components of the permeability tensor. Equations for the component velocities are based on a particular generalized form of the Darcy equation. An equation showing a relation between the head observed in fresh water overlying diffused water and the elevation of the contact between fresh water and diffused water is given. The equation is based on the concept of environmental head. It is found to be a suitable basis for defining the specific limitations of the Ghyben-Herzberg and the Hubbert equations when they are used for fresh-diffused-salt water environments.

INTRODUCTION

The purpose of this paper is (1) to introduce the concept of environmental-water head and to define its relation to point-water and fresh-water heads; (2) to state and illustrate equations for determining hydraulic gradients and also rates and directions of flow from environmental-water heads along the vertical and from the fresh-water heads along the horizontal in ground water of variable density; and (3) to illustrate an equation based on environmental-water heads by means of which the specific limitations of the Ghyben-Herzberg and the Hubbert equations can be defined.

The symbols are explained as they appear in the presentation. A list of all symbols used in this paper is in Appendix 1.

1. POINT-WATER, FRESH-WATER, AND ENVIRONMENTAL-WATER HEADS

In this section, point-water, fresh-water, and environmental-water heads are defined, and equations for determining hydraulic gradients and vector velocities by use of fresh-water and environmental-water heads are stated and illustrated.

Water at a point in ground water of variable density is called point water. Point water may be fresh, diffused, or salt. The head at any point varies with the datum and the kind of water used in the well for measuring the head.

Point-water head at a point in ground water of variable density is defined as the water level, referred to a given datum,[1] in a well filled sufficiently with the water of the type at the point to balance the existing pressure at the point. From this definition (Fig. 1*a*),

$$\rho_i H_{ip} = Z_i \rho_i + p_i/g \qquad (1)$$

where

i = any point in ground water of variable density.
ρ_i = density of water at i.
H_{ip} = point-water head at i.
Z_i = elevation of i, measured positively upward.
p_i = pressure at i.
g = gravitational acceleration.

The first subscript in H_{ip} refers to the point in

[1] All heads and elevations in this paper are referred to mean-sea-level datum.

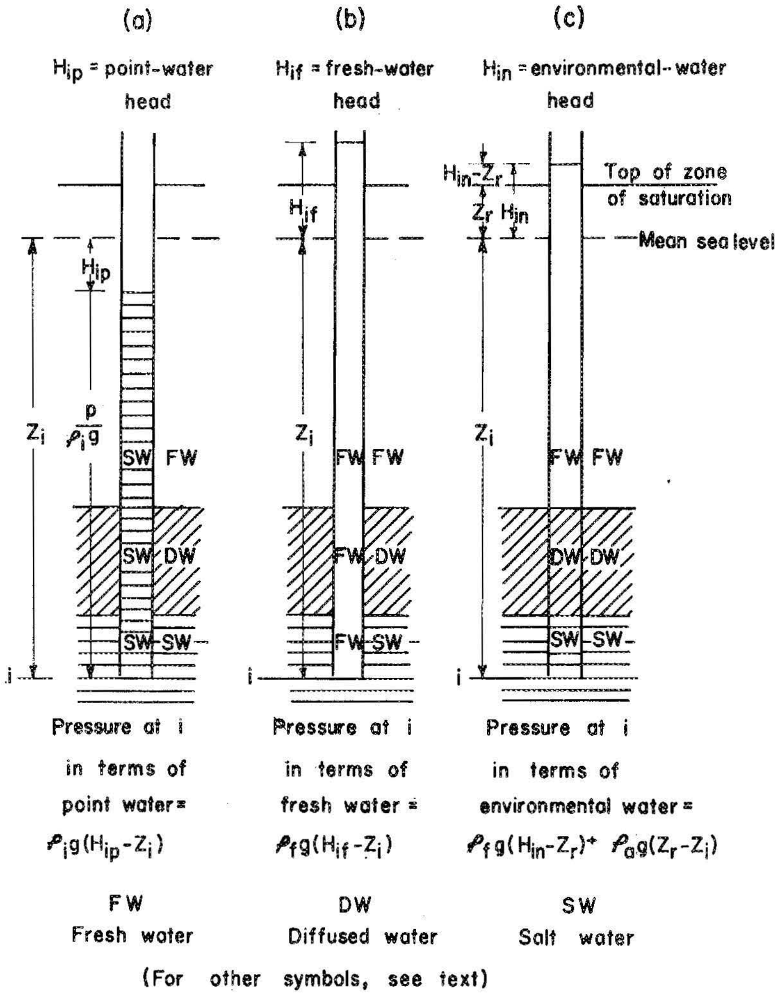

Fig. 1. Heads in ground water of variable density.

question and the second subscript signifies that the water in the well is that at the point.

Fresh-water head at any point i in ground water of variable density is defined as the water level in a well filled with fresh water from i to a level high enough to balance the existing pressure at i. The pressure at i is given by $\rho_i g(H_{ip} - Z_i)$ when point water at i is used in the well. It is given by $\rho_f g(H_{if} - Z_i)$ when fresh water is used (Fig. 1b). From this equality, the fresh-water head may be expressed in terms of the point-water head as

$$\rho_f H_{if} = \rho_i H_{ip} - Z_i(\rho_i - \rho_f) \qquad (2)$$

where

ρ_f = density of fresh water.
H_{if} = fresh-water head at i.

Environmental water between a given point in a ground-water system and the top of the zone of saturation is herein defined to be the water of constant or variable density occurring in the environment along a vertical between that point and the top of the zone of saturation. Environmental-water head at a given point in ground water of variable density is defined as a fresh-water head reduced by an amount corresponding to the difference of salt mass in fresh water and that in the environmental water between that point and the top of the zone of saturation. The reduction is occasioned, in effect, by the replacement of the fresh water in the well shown in Figure 1*b* by the diffused water and salt water as shown in Figure 1*c* at depths where diffused water and salt water occur in the environment.

Environmental-water head, as determined from the equality of pressures expressed in terms of environmental water and fresh water, is

$$\rho_f H_{in} = \rho_f H_{if} - (\rho_f - \rho_a)(Z_i - Z_r) \qquad (3)$$

where

ρ_a = average density of water between Z_r and i, as defined by

$$\frac{1}{Z_r - Z_i} \int_{Z_i}^{Z_r} \rho \, dz.$$

H_{in} = environmental-water head at i.

Z_r = elevation of reference point from which the average density of water to i is determined and above which water is fresh; elevation measured positively upward.

It can also be determined from the equality of pressures expressed in terms of environmental water (Fig. 1*c*) and point water (Fig. 1*a*) as

$$\rho_f H_{in} = \rho_i H_{ip} - Z_i(\rho_i - \rho_a) - Z_r(\rho_a - \rho_f) \qquad (4)$$

In (3) the relation between environmental-water head and fresh-water head is expressed. The relation between environmental-water head and point-water head is expressed in (4).

As stated, ρ_a is the average density of water between a selected reference point and the point in question. The reference point must be the top of the zone of saturation if the uppermost water body is not fresh. If the uppermost body is fresh, any reference point may be selected between the top of the zone of saturation and the first contact of fresh water with diffused water. When the reference point can be and is made coincident with the datum (mean sea level), $Z_r = 0$, and (3) and (4) reduce to

$$\rho_f H_{in} = \rho_f H_{if} - Z_i(\rho_f - \rho_a) \qquad (3a)$$

and

$$\rho_f H_{in} = \rho_i H_{ip} - Z_i(\rho_i - \rho_a) \qquad (4a)$$

In these equations, ρ_a is then the average density of water between mean sea level and point i.

The use of the contact of fresh water with the underlying diffused water as a reference point sets up (3) and (4) in a convenient form when this contact is of specific interest to the problem (see equation 9).

Hydraulic gradients. The gradient of (1) for any point i in ground water of variable density is

$$\nabla p_i/g + \rho_i \mathbf{k} = \nabla(\rho_i H_{ip}) - Z_i \nabla \rho_i \qquad (5a)$$

where

∇ = gradient operator.

$\mathbf{k}$ = unit vector directed upward along a vertical.

Similarly, from (1) and (2) with H_{ip} eliminated, we obtain

$$\nabla p_i/g + \rho_i \mathbf{k} = \rho_f \nabla H_{if} + (\rho_i - \rho_f)\mathbf{k} \qquad (5b)$$

Also from (3), keeping in mind that $(Z_i - Z_r)\,\partial(\rho_a)/\partial z = (\rho_i - \rho_a)$, we obtain

$$\rho_f \nabla H_{in} - (Z_i - Z_r)\left[\frac{\partial(\rho_a)}{\partial x}\,\mathbf{i} + \frac{\partial(\rho_a)}{\partial y}\,\mathbf{j}\right] = \rho_f \nabla H_{if} + (\rho_i - \rho_f)\mathbf{k} \qquad (5c)$$

where $\mathbf{i}$, $\mathbf{j}$ = unit vectors normal to one another in a horizontal plane. Equations 5*a* to 5*c* represent gradients determined from point-water densities and pressures, from point-water densities and point-water heads, from fresh-water density and fresh-water heads, or from fresh-water density and environmental-water heads.

Equation 5*c* is of special value to this paper because of the following. Along a vertical, the left-hand side of (5*c*) reduces conveniently to $\rho_f(\partial H_{in}/\partial z)$, which is simply a product of fresh-water density and a gradient defined by environmental-water heads. Along any horizontal, the right-hand side of (5*c*) reduces to $\rho_f(\partial H_{if}/\partial x)$, which is simply a product of a fresh-water density and a gradient defined by fresh-water heads.

Because environmental-water heads define hydraulic gradients along a vertical, they are comparable along a vertical. This is evidently not the case for point-water or fresh-water heads. Also, because fresh-water heads define hydraulic gradients along a horizontal in ground water of variable density, they are comparable along a horizontal. This is not the case for point-water or environmental-water heads.

Velocity components. A vector velocity is the rate and direction of flow. A variation of the

Darcy equation expressing vector velocity for steady flow at a point in a ground-water system is

$$\mathbf{q}_i = -\frac{k_i}{\mu_i} g\left[\frac{\nabla p_i}{g} + \rho_i \mathbf{k}\right] \qquad (6)$$

where

$\mathbf{q}$ = vector velocity at i.
k_i = permeability of the medium at i.
μ_i = dynamic viscosity at i.

The proper value of permeability depends on the medium. In an isotropic medium the permeability is constant in all directions. In an anisotropic medium the permeability varies with direction and is a tensor. The permeability tensor has three principal directional permeabilities according to the tensor theory of permeability substantiated by *Scheidegger* [1957, pp. 47–66]. He summarized the results of studies on directional permeabilities in anisotropic media made by him and other investigators. He referred to Ferrandon and Litwiniszyn, who developed more-or-less identical theories on the permeability tensor and on flow through anisotropic media. The theories state in effect that the vector velocity may be computed from (6) using a symmetric permeability tensor with components K_{rs}.

For the bracketed term in (6), any one of its equivalents in (5*a-c*) may be used with components K_{rs} of the permeability tensor to define velocity components in ground water of variable density. The use of (5*c*) in (6) is suited for the definition of specific component velocities from environmental-water and fresh-water heads as indicated in the two following paragraphs.

For the case of random orientation of coordinates in relation to the axes of principal directional permeabilities, components of velocity in (6) may be expressed in terms of the components K_{rs} of the permeability tensor and in terms of gradients of environmental-water and fresh-water heads from (5*c*), as follows:

$$v_x = -K_{11}\frac{g}{\mu_i}\left[\rho_f \frac{\partial H_{if}}{\partial x}\right] - K_{12}\frac{g}{\mu_i}\left[\rho_f \frac{\partial H_{if}}{\partial y}\right] - K_{13}\frac{g}{\mu_i}\left[\rho_f \frac{\partial H_{in}}{\partial z}\right] \qquad (7a)$$

$$v_y = -K_{21}\frac{g}{\mu_i}\left[\rho_f \frac{\partial H_{if}}{\partial x}\right] - K_{22}\frac{g}{\mu_i}\left[\rho_f \frac{\partial H_{if}}{\partial y}\right] - K_{23}\frac{g}{\mu_i}\left[\rho_f \frac{\partial H_{in}}{\partial z}\right] \qquad (7b)$$

$$v_z = -K_{31}\frac{g}{\mu_i}\left[\rho_f \frac{\partial H_{if}}{\partial x}\right] - K_{32}\frac{g}{\mu_i}\left[\rho_f \frac{\partial H_{if}}{\partial y}\right] - K_{33}\frac{g}{\mu_i}\left[\rho_f \frac{\partial H_{in}}{\partial z}\right] \qquad (7c)$$

where v_x, v_y, and v_z are velocity components along the x, y, and z coordinates, respectively, and where the K_{rs} components are as indicated in Appendix 2.

When the x and y coordinates are horizontal, the z coordinate is vertical, and when the coordinates coincide with the axes of the principal directional permeabilities,

$$v_x = -K_{11}\frac{g}{\mu_i}\left[\rho_f \frac{\partial H_{if}}{\partial x}\right] \qquad (8a)$$

$$v_y = -K_{22}\frac{g}{\mu_i}\left[\rho_f \frac{\partial H_{if}}{\partial y}\right] \qquad (8b)$$

and

$$v_z = -K_{33}\frac{g}{\mu_i}\left[\rho_f \frac{\partial H_{in}}{\partial z}\right] \qquad (8c)$$

where K_{11}, K_{22}, K_{33} are, in this case, the principal directional permeabilities in the x, y, z directions.

Thus, in an anisotropic medium with horizontal beds, the components of velocity in the two horizontal directions corresponding to the axes of the two principal directional permeabilities may be computed from (8*a*) and (8*b*) by using in each case the appropriate directional permeability, gravity, viscosity of water, density of fresh water, and the hydraulic gradient defined by fresh-water heads. The vertical component of velocity may be computed from (8*c*) by using the principal directional permeability along the vertical, gravity, viscosity of water, density of fresh water, and the hydraulic gradient defined by environmental-water heads.

Because the permeability in an anisotropic medium is a tensor, the direction of flow is usually not parallel to the hydraulic gradient. The direction of flow is defined by the resultant of velocity components defined by (7*a-c*) or (8*a-c*).

Example. The use of environmental-water heads and fresh-water heads for defining hydraulic gradients and velocities at points in an aquifer having fresh, diffused, and salt water is to be illustrated by data furnished through the courtesy of the Government Institute of Water Supply in the Netherlands. The Institute supplied information on water levels and chloride concentrations for July 1957 at suites of wells, each screened at a different depth at each of seven observation and sampling stations in the dune area near The Hague (Fig. 2). The stations are in the general vicinity of infiltration canals and ponds by means of which imported river water is recharged to the ground-water system. Also, ground water is pumped from several nearby areas. The geologic environment consists of dune sand above sea level, a clay layer at about sea level, fine to coarse sand zones to depths of about 50 m below sea level, and then fine and silty sand interspersed with layers and lenses of clay below sea level to depths of about 100 m below sea level. These deposits are anisotropic and essentially horizontal.

The data are used in this paper only to illustrate the use of environmental-water and fresh-water heads for determining hydraulic gradients and velocities in water of variable density. Actually, the computed heads, gradients, directions, and velocities may not be precise because (*a*) the point-water heads and chlorides were not observed on the same day in July 1957, (*b*) heads are reportedly accurate only to the nearest 0.02 m, and (*c*) densities were computed on the basis of 1.025 g/ml for a water with 18,000 ppm chlorides. Two-dimensional steady-state flow was also assumed.

Point-water, fresh-water, and environmental-water heads are given in Figure 2. Fresh-water heads were determined by (2) and environmental-water heads by (3) or (3*a*).

Point-water heads in fresh water are also envirionmental-water heads and as such compare directly with environmental-water heads in the diffused water along the verticals. The environmental-water heads at the seven stations indicate components of flow upward at some points and downward at other points in the fresh water as well as in the diffused water.

Point-water heads in fresh water are also fresh-water heads, and as such compare with fresh-water heads in the diffused water along the horizontals. Fresh-water heads at the seven stations indicate components of flow in a landward direction at most of the points in the fresh water as well as in the diffused water.

Consider the point at the −54-m elevation at station MN. A hydraulic gradient of about 0.001 in favor of a downward component of flow is computed from the tangent at the point in question to the environmental-water head vs. depth curve. Also a gradient of about 0.0015 in favor of a landward component of flow is computed from the tangent at the point in question to the fresh-water head vs. distance curve. (A fresh-water head of 1.12 m at the −54-m elevation at station N was used for this purpose. It was determined by (4*a*) from an environmental-water head interpolated between that of 1.03 m at the −47-m elevation and 1.04 m at the −56-m elevation.)

If we compare the direction of flow with that of the resultant hydraulic gradient at the −54-m elevation at station MN and assume that the horizontal and vertical permeabilities at the point are the same as the principal directional permeabilities, then the direction flow may be determined from the horizontal and vertical velocity components computed by (8*a*) and (8*c*), respectively, by using the horizontal and vertical permeabilities and the gradients defined by fresh-water and environmental-water heads. For an assumed ratio of horizontal to vertical permeability of 5, the direction of flow at the −54-m elevation at station MN would be landward and downward at an angle of 8° from the horizontal. For an assumed ratio of 10, the direction would be 4° from the horizontal. If these ratios were of the right order of magnitude, the direction of flow at the point would be landward and essentially horizontal. In comparison, the resultant hydraulic gradient was landward and downward at an angle of about 34° from the horizontal.

The horizontal permeability at the −54-m elevation is about 25 millidarcys (1 millidarcy = 0.001 cm/sec). This is estimated from the range of values furnished by the Institute. On the basis of this permeability, a horizontal gradient

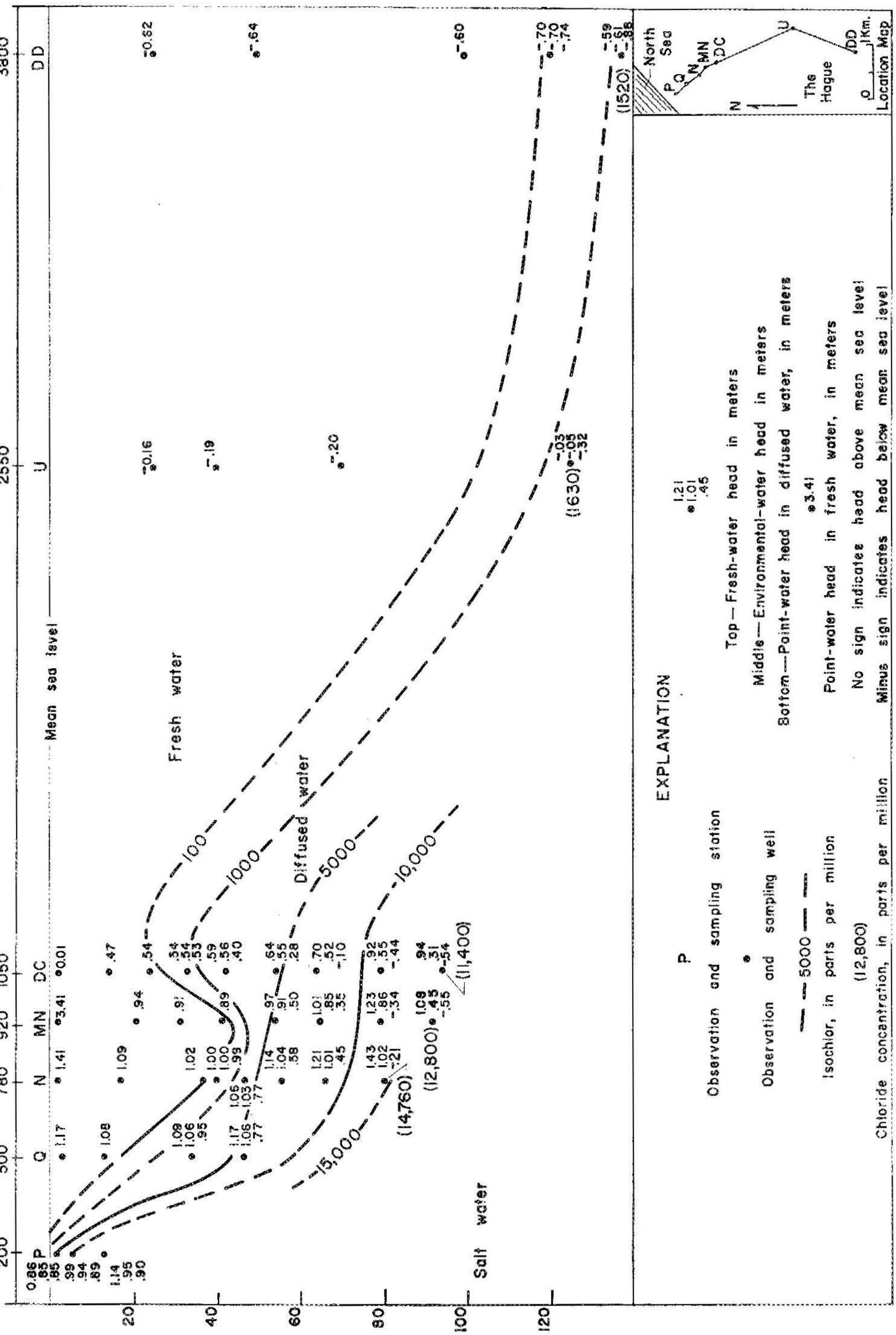

Fig. 2. Heads in coastal ground-water system near The Hague, Netherlands, along section P-DD.

of 0.0015, and a porosity of ⅓, the component of actual velocity[2] along the horizontal would be landward at about 9.7 cm/day (about 0.3 ft/day) at the −54-m elevation in station MN. On the basis of a horizontal to vertical ratio of 5 to 10, a vertical gradient of 0.001, and a porosity of ⅓, the component of actual velocity[3] along the vertical would be downward at about 1.3 to 0.6+ cm/day (about 0.04 to 0.02 ft/day).

2. Equation 9 vs. Ghyben-Herzberg and Hubbert Equations

In this section an equation based on the concept of environmental-water head is used for defining the specific limitations of the Ghyben-Herzberg and Hubbert equations when they are used for fresh-diffused-salt water environments.

Let h denote the difference between the environmental-water head at any point 1 in fresh water and that at any point 2 in salt water along a vertical in a ground-water system having fresh water, diffused water, and salt water (top to bottom). Actually h represents the head loss due to vertical velocities between points 1 and 2. From (4), written for points 1 and 2, we get

$$\rho_1 H_{1p} = \rho_1 h + \rho_2 H_{2p} - Z_2(\rho_2 - \rho_a) - Z_d(\rho_a - \rho_1) \qquad (9)$$

where $h = H_{1n} - \mathrm{H}_{2n} = H_{1p} - \mathrm{H}_{2n}$ and where Z_d is the elevation of the contact of fresh water with diffused water and also of the reference point from which ρ_a is computed. A derivation of (9) from the Darcy equation is given in Appendix 3.

Equation 9 may be interpreted as a relation between H_{ip}, a point-water or fresh-water head in fresh water, and Z_d, the elevation of the contact of fresh water with diffused water. The relation includes a term which accounts for the difference in environmental-water head between points 1 and 2, a term which accounts for the point-water head in salt water, and two terms which account for the variable density in the zone of diffusion.

Using the symbols of this paper, the Ghyben-Herzberg equation [*Ghyben*, 1889; *Herzberg*, 1901] is

$$\rho_1 H_{1p} = -Z_d'(\rho_2 - \rho_1) \qquad (10)$$

[2] v_x from (8*a*) divided by the porosity.

[3] v_z from (8*c*) divided by the porosity.

where H_{ip} is the water table and Z_d' is an approximate depth to diffused water. Thus (10) is evidently a special case of (9) in which $h = 0$ and $\rho_a = \rho_2$. Then $Z_d = Z_d'$. Therefore the depth to diffused water computed by the Ghyben-Herzberg equation is correct or approximately correct when (*a*) the difference in environmental-water head between points 1 and 2 is zero or relatively small, (*b*) the point-water head in salt water is zero or relatively small, and (*c*) the zone of diffusion is of zero or relatively small thickness. If only the (*a*) and (*b*) conditions are met, Z_d' is a depth to an indefinite point in the zone of diffusion.

In symbols of this paper, *Hubbert*'s [1940] equation 189 is

$$\rho_1 H_{1p} = \rho_2 H_{2p} - Z_d''(\rho_2 - \rho_1) \qquad (11)$$

This equation expresses the correct relation between Z_d'', the contact or interface between two immiscible liquids ρ_1 and ρ_2 (for $\rho_2 > \rho_1$), and H_{1p} and H_{2p} on the ρ_1-side and ρ_2-side, respectively, of a given point on the interface of the two liquids. Equation 11 would define the contact between fresh water and salt water with a sharp interface. It was used by *Perlmutter, Geraghty, and Upson* [1959] for determining an elevation of such a theoretical contact of fresh water with salt water. The computed elevation was shown to be within the diffused water in an actual fresh-diffused-salt water environment. For the computations, the heads observed at points some distance apart in a vertical were used; one point was in fresh water and the other in salt water. Equation 11 is valid when the head loss due to vertical velocities between the points of observation is negligible. Then at least an approximate elevation of the theoretical contact of fresh water with salt water is determined.

The head loss between any two points in a vertical is actually the difference in environmental heads between the two points. It is defined by the $\rho_1 h$ term in (9). When not negligible, this term should be included with (11) as in the following:

$$\rho_1 H_{1p} = \rho_1 h + \rho_2 H_{2p} - Z_d'''(\rho_2 - \rho_1) \quad (12)$$

Equation 12 is then a special case of (9) in which $\rho_a = \rho_2$ and $Z_d = Z_d'''$. It may be interpreted as a relation between H_{1p}, a point-water head in fresh water, and Z_d''', the actual elevation of the theoretical contact between fresh water and salt water.

Examples. Compare the Z_d elevations computed by (10–12) using the following data for two selected points at station N (Fig. 2) near The Hague in the Netherlands:

$$\rho_1 = 1.000 \text{ g/ml}$$

$$\rho_2 = 1.0205 \text{ g/ml}$$

$$H_{1p} = 1.09 \text{ m}$$

$$H_{2p} = -.21 \text{ m}$$

$$Z_1 = -17 \text{ m}$$

$$Z_2 = -80 \text{ m}$$

Sufficient information was obtained at six additional points in the vertical (Fig. 2) to define the contact between fresh water and diffused water at −37 m and to compute $h = +0.07$ m between points 1 and 2.

$Z_d' = -53$ m was computed by the Ghyben-Herzberg equation (10), $Z_d'' = -64$ m by the Hubbert equation (11), and $Z_d''' = -60$ m by (12). The difference of −11 m between Z_d' and Z_d'' is attributed to the term $\rho_2 H_{2p}$; the difference of +4 m between Z_d'' and Z_d''' is due to the +0.07-m difference in environmental heads at points 1 and 2. Because the corrections in this case are small, the Ghyben-Herzberg and Hubbert equations give elevations within the zone of diffusion. Also these elevations are not appreciably different from the actual elevation of the theoretical contact, Z_d''', between fresh water and salt water. The theoretical contact is 23 m below the contact of fresh water with diffused water.

The H_{1p} at the −17-m elevation was selected arbitrarily for the above. If, instead, the $H_{1p} = -1.41$ m at the −2-m elevation (Fig. 2) is selected, then $Z_d' = -69$ m from (1), $Z_d'' = -79$ m from (11), and $Z_d''' = -60$ m from (12). In this case Z_d''' is the same as computed previously. (In the computation, $h = +0.39$ m was used.) However, the Z_d' by the Ghyben-Herzberg equation is 9 m lower than Z_d''', and the Z_d'' by the Hubbert equation is practically at point 2 in salt water.

Compare also the depths Z_d, computed from (10–12) using the following data obtained in October 1958 at two wells screened at two different depths in the same vertical at a site in Cedarhurst, L. I., N. Y.:

$$\rho_1 = 0.999 + \text{ g/ml}$$

$$\rho_2 = 1.020 + \text{ g/ml}$$

$$H_{1p} = 3.57 \text{ ft}$$

$$H_{2p} = -4.72 \text{ ft}$$

$$Z_1 = -167 \text{ ft}$$

$$Z_2 = -520 \text{ ft}$$

The average density at Cedarhurst in the zone of diffusion between the −320-ft and −490-ft elevations is computed to be about 1.010 g/ml. On this basis, h = −0.47 ft is determined from (9).

$Z_d' = -170$ ft is obtained by the Ghyben-Herzberg equation, $Z_d'' = -399$ ft by the Hubbert equation, and $Z_d''' = -421$ ft by (12). The difference of −229 ft between Z_d' and Z_d'' is the result of the $\rho_2 H_{2p}$ term. In this case, the Ghyben-Herzberg equation gives an elevation which is not even in the zone of diffusion and is as much as 150 ft above it. The Hubbert equation gives a depth 22 ft shallower than the actual elevation of the theoretical contact between fresh water and salt water. The 22 ft difference indicates that every 0.1-ft difference in environmental-water head between points 1 and 2 makes a difference of nearly 5 ft between the depth computed by the Hubbert equation and that computed by (12).

The theoretical contact of fresh water with salt water at Cedarhurst is about 101 ft below the contact of fresh water with diffused water, defined by electrical log to be at −320 ft.

Elevations computed by equations 9 to 12. Z_d in (9) is the elevation of the contact of fresh water with diffused water. The relation in (9) between point-water head in fresh water and the depth to this contact is not simple and direct; it depends on several variables. However, the elevation of the contact can be computed if there is a suitable basis for estimating h, H_{2p}, and ρ_a for a given Z_2 and ρ_2.

Z_d''' in (12) is the elevation to the theoretical contact between fresh water and salt water. It can be computed from a point-water head in fresh water if there is a suitable basis for estimating h and H_{2p} for a given Z_2 and ρ_2.

As shown, the Z_d' computed by the Ghyben-

Herzberg equation may not necessarily be an elevation of a point in the zone of diffusion. Also as shown, the Z_d'' computed by the Hubbert equation may not necessarily be an elevation of a point in the zone of diffusion. However, the Ghyben-Herzberg and Hubbert equations can yield elevations of points in the zone of diffusion for conditions approximating those stated for (10) and (11).

Appendix 1. Symbols Used in This Paper

g = gravitational acceleration.
$\mathbf{g}$ = gravitational acceleration vector.
H_{ip}, H_{if}, H_{in} = point-water head, fresh-water head, and environmental-water head at i, respectively.
Z_r = elevation of reference point, measured positively upward.
ρ_i = density of water at i.
ρ_a = average density of water between depths Z_r and i.
ρ_f = density of fresh water.
μ_i = viscosity of water at i.
$\partial h_{if}/\partial x$ = hydraulic gradient at i in x direction (defined by fresh water heads).
$\partial h_{if}/\partial y$ = hydraulic gradient at i in y direction (defined by fresh-water heads).
$\partial h_{in}/\partial z$ = hydraulic gradient at i and z direction (defined by environmental-water heads).
∇ = gradient operator.

Appendix 2. K_{rs} Components of Permeability Tensor

$$\begin{bmatrix} K_{11} = \begin{Bmatrix} K_1 l_1 l_1 \\ +K_2 l_2 l_2 \\ +K_3 l_3 l_3 \end{Bmatrix} & K_{12} = \begin{Bmatrix} K_1 l_1 m_1 \\ +K_2 l_2 m_2 \\ +K_3 l_3 m_3 \end{Bmatrix} & K_{13} = \begin{Bmatrix} K_1 l_1 n_1 \\ +K_2 l_2 n_2 \\ +k_3 l_3 n_3 \end{Bmatrix} \\ K_{21} = \begin{Bmatrix} K_1 l_1 m_1 \\ +K_2 l_2 m_2 \\ +K_3 l_3 m_3 \end{Bmatrix} & K_{22} = \begin{Bmatrix} K_1 m_1 m_1 \\ +K_2 m_2 m_2 \\ +K_3 m_3 m_3 \end{Bmatrix} & K_{23} = \begin{Bmatrix} K_1 m_1 n_1 \\ +K_2 m_2 n_2 \\ +K_3 m_3 n_3 \end{Bmatrix} \\ K_{31} = \begin{Bmatrix} K_1 l_1 n_1 \\ +K_2 l_2 n_2 \\ +K_3 l_3 n_3 \end{Bmatrix} & K_{32} = \begin{Bmatrix} K_1 m_1 n_1 \\ +K_2 m_2 n_2 \\ +K_3 m_3 n_3 \end{Bmatrix} & K_{33} = \begin{Bmatrix} K_1 n_1 n_1 \\ +K_2 n_2 n_2 \\ +K_3 n_3 n_3 \end{Bmatrix} \end{bmatrix}$$

h = difference in environmental-water heads at point 1 in fresh water and point 2 in diffused or salt water.
i = point in variable density ground water; i = 1, point in fresh water; i = 2, point in diffused or salt water.
$\mathbf{i}$ = unit vector directed along a horizontal.
k_i = permeability of the medium at i.
$\mathbf{k}$ = unit vector directed upward along a vertical.
P_i = pressure at i.
$\mathbf{q}_i$ = vector velocity at i.
v_x, v_y, v_z = component of velocity along axis x, y, z, respectively.
Z_d, Z_d', Z_d'', Z_d''' = elevation to diffused water in (9–12), respectively.
Z_i = elevation of i, measured positively upward.

where

K_1, K_2, K_3 = principal directional permeabilities.
l_1, m_1, n_1 = directional cosines between K_1 and x, y, z coordinates.
l_2, m_2, n_2 = directional cosines between K_2 and x, y, z coordinates.
l_3, m_3, n_3 = directional cosines between K_3 and x, y, z coordinates.

Appendix 3

The introduction and use of the 'environmental-water head' is a valid interpretation of the Darcy equation. This was indicated by Professor R. Skalak, who reviewed the early draft of the paper. The following is an excerpt from his written communication:

The generalized Darcy equation in vector form is

$$\mathbf{q} = -\frac{k}{\mu}(\nabla p - \rho_i \mathbf{g}) \qquad (A)$$

The z component of this equation may be written as

$$v\frac{\mu}{k\rho_1 g} = -\frac{1}{\rho_1 g}\left(\frac{\partial p}{\partial z} + \rho_i g\right) \qquad (B)$$

where

v = vertical component of velocity.
μ = viscosity.
k = permeability.
$\mathbf{g}$ = gravitational acceleration vector.
ρ_1 = density of fresh water.
p = pressure.
z = vertical coordinate, plus upwards.
ρ_i = density of fluid at any point.

It appears that the right-hand side of (B) is the same as the 'gradient of environmental head' that you have defined. The left-hand side is the head loss gradient in terms of velocity and is essentially the gradient of h in your equation 9. To show the equivalence of the above to your (9), integrate (B) from point 2 to point 1.

$$\int_{z_2}^{z_1} v\frac{\mu}{k\rho_1 g}\,dz = -\int_{z_2}^{z_1}\frac{1}{\rho_1 g}\left(\frac{\partial p}{\partial z} + \rho_i g\right)\partial z \qquad (C)$$

or

$$\int_{z_2}^{z_1}\frac{v\mu}{kg}\,dz = -\frac{1}{g}[p_1 - p_2] - \int_{z_2}^{z_r}\rho_i\,dz - \int_{z_r}^{z_1}\rho_i\,dz \qquad (D)$$

Now define h to be the head loss due to vertical velocities, i.e., such that $-\rho_1 h$ is equal to the left-hand side of (D). Further define ρ_a such that

$$\int_{z_2}^{z_r}\rho_i\,dz = \rho_a(z_r - z_2) \qquad (E)$$

This is the same as your ρ_a. Further note that

$$p_1 = \rho_1 g(H_{1p} - z_1)$$

and

$$p_2 = \rho_2 g(H_{2p} - z_2) \qquad (F)$$

Substituting in (D):

$$-\rho_1 h = -\rho_1(H_{1p} - z_1) + \rho_2(H_{2p} - z_2) - \rho_a(z_r - z_2) - \rho_1(z - z_r) \qquad (G)$$

The right-hand side of (G) is $(-\rho_1 H_{1n} + \rho_1 H_{2n})$ where H_{1n} and H_{2n} are the environmental heads defined as you have suggested. Equation (G) may be written as

$$\rho_1 H_{1p} = \rho_1 h + \rho_2 H_{2p} - z_2(\rho_2 - \rho_a) - z_r(\rho_a - \rho_1) \qquad (H)$$

which is exactly your (9), when $Z_r = Z_d$.

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